SPECTRAL ANALYSIS OF THE FRIEDMAN-KELLER EQUATIONS OF HOMOGENEOUS TURBULENCE

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A problem of Cauchy type is considered for the Friedman-Keller equations in the case of homogeneous turbulence. The spectral tensors, which are the generalized Fourier transforms of the correlation tensors, are represented at an arbitrary instant of time in the form of the sum of a series of multiple integrals of the spectral tensors of the initial distribution function.

Let $u_{a_k}(M_k)$ be the turbulent velocity at the point M_k; then we denote the velocity correlation tensor by

$$T^{(n)}_{a_0 a_1, \ldots, a_{n-1}}(M_0, M_1, \ldots, M_{n-1}) = \langle u_{a_0}(M_0)\, u_{a_1}(M_1), \ldots, u_{a_{n-1}}(M_{n-1})\rangle$$

the spectral velocity tensor by

$$\tau^{(n)}_{a_0 a_1, \ldots, a_{n-1}}(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{n-1}) = \frac{1}{(2\pi)^{3(n-1)}} \int T^{(n)}_{a_0 a_1, \ldots, a_{n-1}} \exp(-i\mathbf{k}_m \mathbf{r}_m)\, dr$$

and the pressure-velocity correlation tensor by

$$P^{(n)}_{a_0 a_1, \ldots, O_k, \ldots, a_{n-1}}(M_0, M_1, \ldots, M_{n-1}) = \langle u_{a_0}(M_0), \ldots, p(M_k), \ldots, u_{a_{n-1}}(M_{n-1})\rangle$$

Here summation is carried out over repeated indices.

In the statistical theory of turbulence the correlation tensors determining the distribution-function satisfy the well-known Friedman-Keller system of equations, derived from the Navier-Stokes equations [1].

For the case of homogeneous turbulence in an incompressiboe fluid they have the form

$$\left\{\frac{\partial}{\partial t}-\nu[\Delta_0+\Delta_1]\right\}T^{(2)}_{a_0a_1}= \tag{1.1}$$

$$=-\frac{\partial}{\partial(\mathbf{x}_0)_{a_2}}\lim_{M_2\to M_0}T^{(3)}_{a_0a_1a_2}-\frac{\partial}{\partial(\mathbf{x}_1)_{a_2}}\lim_{M_2\to M_1}T^{(3)}_{a_0a_1a_2}-\frac{1}{\rho}\frac{\partial}{\partial(\mathbf{x}_0)_{a_0}}P^{(2)}_{0_0a_1}-\frac{1}{\rho}\frac{\partial}{\partial(\mathbf{x}_1)_{a_1}}P^{(2)}_{a_00_1}$$

. . . .

$$\left\{\frac{\partial}{\partial t}-\nu\sum_{k=0}^{n-1}\Delta_k\right\}T^{(n)}_{a_0,\ldots,a_{n-1}}=$$

$$=-\sum_{k=0}^{n-1}\frac{\partial}{\partial(\mathbf{x}_k)_{a_n}}\lim_{M_n\to M_k}T^{(n+1)}_{a_0a_1,\ldots,a_n}-\frac{1}{\rho}\sum_{k=0}^{n-1}\frac{\partial}{\partial(\mathbf{x}_k)_{a_k}}P^{(n)}_{a_0,\ldots,0_k,\ldots,a_{n-1}} \tag{1.n}$$

We introduce homogeneous coordinates

$$\frac{\partial}{\partial x_{a_0}}=-\sum_{k=1}^{n-1}\frac{\partial}{\partial r_{a_k}},\qquad \frac{\partial}{\partial x_{a_k}}=\frac{\partial}{\partial r_{a_k}}$$

and the dimensionless variables u_0, the mean square velocity, and l_2, l_3, ..., l_n, characteristic scales for the correlation moments of order 2, 3, ..., n.

Then eliminating the pressure by means of the relation

$$\frac{1}{\rho}p(\mathbf{x})=\frac{1}{4\pi}\int\frac{\partial^2u_i'\,u_j'}{\partial x_i'\,\partial x_j'}\frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$$

we put the system (1.1) (1.n) into the characteristic form for the theory of homogeneous turbulence:

$$\begin{aligned}\left\{\frac{\partial}{\partial t}-\frac{2}{R_2}\Delta_{\mathbf{r}_1}\right\}T^{(2)}_{a_0a_1}&=-\frac{\partial}{\partial(\mathbf{r}_1)_{a_2}}(\lim_{M_2\to M_1}\mathrm{T}^{(3)}_{a_0a_1a_2}-\lim_{M_2\to M_0}T^{(3)}_{a_0a_1a_2})-\\&-\frac{1}{4\pi}\left(\frac{\partial}{\partial(\mathbf{r}_1)_{a_1}}\int\frac{1}{|\mathbf{r}_1-\mathbf{s}|}\frac{\partial^2}{\partial(\mathbf{s})_{a_1}\partial(\mathbf{s})_{a_2}}\lim T^{(3)}_{a_0a_1a_2}\,d\mathbf{s}-\right.\\&\left.-\frac{\partial}{\partial(\mathbf{r}_1)_{a_0}}\int\frac{1}{|\mathbf{r}_1-\mathbf{s}|}\frac{\partial^2}{\partial(\mathbf{s})_{a_0}\partial(\mathbf{s})_{a_2}}\lim T^{(3)}_{a_0a_1a_2}d\mathbf{s}\right)\end{aligned} \tag{2.1}$$

. . . .

$$\begin{aligned}&\left\{\frac{\partial}{\partial t}-\frac{1}{R_n}\left[\sum_{k=0}^{n-1}\Delta_{\mathbf{r}_k}+\left(\sum_{k=0}^{n-1}\nabla_{\mathbf{r}_k}\right)^2\right]\right\}T^{(n)}_{a_0a_1,\ldots,a_{n-1}}=\\&=-\sum_{k=1}^{n-1}\frac{\partial}{\partial(\mathbf{r}_k)_{a_n}}(\lim_{M_n\to M_k}T^{(n+1)}_{a_0a_1,\ldots,a_n}-\lim_{M_n\to M_0}T^{(n+1)}_{a_0a_1,\ldots,a_n})+\\&+\frac{1}{4\pi}\left(\sum_{k=1}^{n-1}\frac{\partial}{\partial(\mathbf{r}_k)_{a_n}}\int\frac{1}{|\mathbf{r}_k-\mathbf{s}|}\frac{\partial^2}{\partial(\mathbf{s})_{a_k}\partial(\mathbf{s})_{a_n}}\lim_{M_n\to M_k}T^{(n+1)}_{a_0a_1,\ldots,a_n}d\mathbf{s}-\right.\\&\left.-\sum_{k=1}^{n-1}\frac{\partial}{\partial(\mathbf{r}_k)_{a_0}}\int\frac{1}{|\mathbf{r}_k-\mathbf{s}|}\frac{\partial^2}{\partial(\mathbf{s})_{a_0}\partial(\mathbf{s})_{a_k}}\lim_{M_n\to M_0}T^{(n+1)}_{a_0a_1,\ldots,a_n}d\mathbf{s}\right)\end{aligned} \tag{2.n}$$

The Reynolds numbers R_n in equations (2.1) (2.n) are different, and according to the physical meaning of correlation, form a decreasing sequence.

The formulation of the spectral analog of the infinite system of Friedman-Keller equations is associated with the difficulty that the Fourier transforms in the classical sense do not exist for the correlation tensors of higher than third order. For example, for $\mathbf{r}_1$, $\mathbf{r}_2 \to \infty$ and $(\mathbf{r}_1 - \mathbf{r}_2)$ and $\mathbf{r}_3$ finite, the fourth-order moment tends to

$$\lim T^{(4)}_{a_0a_1a_2a_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = T^{(3)}_{a_0a_1}(\mathbf{r}_1 - \mathbf{r}_2)\, T^{(2)}_{a_2a_3}(\mathbf{r}_3) \neq 0$$

We therefore introduce the generalized Fourier transform [2, Appendix] of the function

$$\Phi\,[f(x)] = \frac{1}{2\pi}\int f(x)\, e^{-ikx}\, dx = \varphi(k) \tag{3}$$

In the final expressions for solutions having physical significance there will appear only integrals of multiplicity $n - 1$ over the wave space of the generalized spectral tensors of rank n, which will be regular functions; the intermediate calculations reduce simply to integral operations; therefore the introduction of these generalized functions is correct.

We show that the Fourier transform

$$\Phi\,(\lim_{M_n \to M_0} T^{(n+1)}_{a_0,\ldots,a_n}) = \int \tau^{(n+1)}_{a_0,\ldots,a_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n)\, d\mathbf{k}_n \tag{4}$$

$$\Phi\,(\lim_{M_n \to M_m} T^{(n+1)}_{a_0,\ldots,a_n}) = \int \tau^{(n+1)}_{a_0,\ldots,a_n}\, \mathbf{k}_1, \ldots, \varkappa_m, \ldots, \mathbf{k}_n)\, d\mathbf{k}_n$$

$$\varkappa_m = -(\mathbf{k}_1 + \ldots + \mathbf{k}_n) \tag{5}$$

The property (4) is easily obtained from the inverse Fourier transformation.

With the transformation of coordinates

$$\rho_i = \mathbf{r}_i - \mathbf{r}_m\, (m \neq i), \qquad \rho_i = -\mathbf{r}_m\, (m = i)$$

we reduce (5) to (4). Here the wave numbers κ_i conjugate to the ρ_i transform as

$$\varkappa_i = \mathbf{k}_i\, (i \neq m), \qquad \varkappa_i = \sum \mathbf{k}_l\, (i = m)$$

According to a well-known formula of the theory of the Fourier integral we have

$$\Phi\left[\int_{-\infty}^{+\infty} v(x-y)\,w(y)\,dy\right] = 2\pi V(k)\,W(k), \qquad V(k) = \Phi(v), \qquad W(k) = \Phi(w) \tag{6}$$

The Fourier transform of the Poisson kernel is

$$\Phi\left[\frac{1}{|\mathbf{r}-\mathbf{s}|}\right] = \frac{1}{k^2}$$

Taking into account the properties (4) and (5) of spectral tensors, we give the spectral formulation of the Friedman-Keller equations for homogeneous turbulence:

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \frac{2}{R_2}\mathbf{k}_1^2\right\}\tau^{(2)}_{a_0a_1} = -i(\mathbf{k}_1)_{a_2}\left[\int \tau^{(3)}_{a_0a_1a_2}(\varkappa_1, \mathbf{k}_2)\,d\mathbf{k}_2 - \right.\\
&\left.-\int \tau^{(3)}_{a_0a_1a_2}(\mathbf{k}_1, \mathbf{k}_2)\,d\mathbf{k}_2\right] + i2\pi^2\left\{(\mathbf{k}_1)_{a_1}\left[\frac{(\mathbf{k}_1)_{a_1}(\mathbf{k}_1)_{a_2}}{\mathbf{k}_1^2}\int \tau^{(3)}_{a_0a_1a_2}(\varkappa_2, \mathbf{k}_2)\,d\mathbf{k}_2\right] - \right.\\
&\left. - (\mathbf{k}_1)_{a_0}\left[\frac{(\mathbf{k}_1)_{a_0}(\mathbf{k}_1)_{a_2}}{\mathbf{k}_1^2}\int \tau^{(3)}_{a_0a_1a_2}(\mathbf{k}_1, \mathbf{k}_2)\,d\mathbf{k}_2\right]\right\}
\end{aligned} \tag{7.1}$$

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \frac{1}{R_3}\left[\mathbf{k}_1^2 + \mathbf{k}_2^2 + (\mathbf{k}_1+\mathbf{k}_2)^2\right]\right\}\tau^{(3)}_{a_0a_1a_2} = \\
&= -i\sum_{m=1}^{2}(\mathbf{k}_m)_{a_3}\left[\int \tau^{(4)}_{a_0a_1a_2a_3}(\mathbf{k}_1, \varkappa_m, \mathbf{k}_3)\,dk_3 - \int \tau^{(4)}_{a_0a_1a_2a_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\,d\mathbf{k}_3\right] + \\
&+ i2\pi^2\sum_{m=1}^{2}\left\{(\mathbf{k}_m)_{a_3}\left[\frac{(\mathbf{k}_m)_{a_m}(\mathbf{k}_m)_{a_3}}{\mathbf{k}_m^2}\int \tau^{(4)}_{a_0a_1a_2a_3}(\mathbf{k}_1, \varkappa_m\ \mathbf{k}_3)\,d\mathbf{k}_3\right] - \right.\\
&\left. - (\mathbf{k}_m)_{a_0}\left[\frac{(\mathbf{k}_m)_{a_0}(\mathbf{k}_m)_{a_3}}{\mathbf{k}_m^2}\int \tau^{(4)}_{a_0a_1a_2a_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\,d\mathbf{k}_3\right]\right\}
\end{aligned} \tag{7.2}$$

. . . .

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \frac{1}{R_n}\left[\sum_{m=1}^{n-1}\mathbf{k}_m^2 + \left(\sum_{m=1}^{n-1}\mathbf{k}_m\right)^2\right]\right\}\tau^{(n)}_{a_0a_1,\ldots,a_{n-1}} = \\
&= -i\sum_{m=1}^{n-1}(\mathbf{k}_m)_{a_n}\left[\int \tau^{(n+1)}_{a_0,\ldots,a_n}(\mathbf{k}_1, \ldots, \varkappa_m, \ldots, \mathbf{k}_n)\,d\mathbf{k}_n - \int \tau^{(n+1)}_{a_0,\ldots,a_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n)\,d\mathbf{k}_n\right] + \\
&+ i2\pi^2\sum_{m=1}^{n-1}\left\{(\mathbf{k}_m)_{a_n}\left[\frac{(\mathbf{k}_m)_{a_m}(\mathbf{k}_m)_{a_n}}{\mathbf{k}_m^2}\int \tau^{(n+1)}_{a_0,\ldots,a_n}(\mathbf{k}_1, \ldots, \varkappa_m, \ldots, \mathbf{k}_n)\,d\mathbf{k}_n\right] - \right.\\
&\left. - (\mathbf{k}_m)_{a_0}\left[\frac{(\mathbf{k}_m)_{a_0}(\mathbf{k}_m)_{a_n}}{\mathbf{k}_m^2}\int \tau^{(n+1)}_{a_0,\ldots,a_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n)\,d\mathbf{k}_n\right]\right\}
\end{aligned} \tag{7.n}$$

The spectral tensors are smooth functions of time, since infinite forces do not exist in a turbulent stream. Regarding (7.1), (7.2), ..., (7.n) as ordinary linear equations with constant coefficients and right-hand sides depending upon time, we solve each of the equations of the system:

$$\tau^{(2)}_{a_0a_1}(\mathbf{k}_1, t) = \tau^{(2)}_{a_0a_1}(\mathbf{k}_1, 0)\,\bar{e}^{P_2t} + \bar{e}^{P_2t}\int_0^t e^{P_2\theta_2} X_2(\tau^{(3)})\,d\theta_2 \tag{8.1}$$

$$\tau^{(3)}_{a_0a_1a_2}(\mathbf{k}_1, \mathbf{k}_2, t) = \tau^{(3)}_{a_0a_1a_2}(\mathbf{k}_1, \mathbf{k}_2, 0)\,\bar{e}^{P_3t} + e^{-P_3t}\int_0^t e^{P_3\theta_3} X_3(\tau^{(4)})\,d\theta_3 \tag{8.2}$$

$$\tau^{(n)}_{a_0,\ldots,a_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}, t) =$$

$$= \tau^{(n)}_{a_0,\ldots,a_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}, 0)\,\bar{e}^{P_nt} + \bar{e}^{P_nt}\int_0^t e^{P_n\theta_n} X_n(\tau^{(n+1)})\,d\theta_n$$

$$P_n \doteq \frac{1}{R_n}\left[\sum_{m=1}^{n-1}\mathbf{k}_m^2 + \left(\sum_{m=1}^{n-1}\mathbf{k}_m\right)^2\right] \tag{8.n}$$

$X_2(\tau^{(3)})$, ..., $X_n(\tau^{(n+1)})$ are the right-hand sides of the equations (7). Or setting

$$e^{-P_nt}\int_0^t e^{P_n\theta_n} X_n(\tau^{(n+1)})\,d\theta_n = L_n(\tau^{(n+1)}), \qquad \tau^{(n)}_{a_0,\ldots,a_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_n, 0) = \tau^{(n)}_0$$

and eliminating $\tau^{(n+1)}$ from $\tau^{(n)}_{a_0,\ldots,a_{n-1}}$, we obtain

$$\tau^{(2)}_{a_0a_1}(\mathbf{k}_1, t) = \tau^{(2)}_{a_0a_1}(\mathbf{k}_1, 0)\,e^{-P_2t} + L_2(\tau^{(3)}_0 e^{-P\theta_3}) + \ldots \tag{9.1}$$
$$\ldots + L_2L_3, \ldots, L_{m-1}(\tau^{(m)}_0 \bar{e}^{-P_m\theta_{m-1}}) + \ldots$$

$$\tau^{(3)}_{a_0a_1a_2}(\mathbf{k}_1, \mathbf{k}_2, t) = \tau^{(3)}_{a_0a_1a_2}(\mathbf{k}_1, \mathbf{k}_2, 0)\,e^{-P_3t} + L_3(\tau^{(4)}_0 e^{-P_4\theta_3}) + \ldots \tag{9.2}$$
$$\ldots + L_3L_4, \ldots, L_{m-1}(\tau^{(m)}_0 e^{-P_m\theta_{m-1}}) + \ldots$$

. . . .

$$\tau^{(n)}_{a_0,\ldots,a_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}, t) = \tau^{(n)}_{a_0,\ldots,a_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}, 0)\,e^{-P_nt} + \tag{9.n}$$
$$+ L_n(\tau^{(n+1)}_0 e^{-P_{n+1}\theta_n}) + \ldots + L_nL_{n+1}, \ldots, L_{m-1}(\tau^{(m)}_0 e^{-P_m\theta_{m-1}}) + \ldots$$

The expressions (9.1), (9.2), (9.n) determine the spectral tensors of the distribution function of a turbulent velocity field at the instant of time $t > 0$ in terms of the spectral tensors of the initial distribution function. The solution obtained is interesting in that it permits carrying out an investigation of turbulence for initial conditions belonging to different types of symmetry.

Appendix. Let the turbulent velocity field possess the distribution function

$$f_{M_0, M_1, \ldots, M_{n-1}}[u_{a_0}(M_0), u_{a_1}(M_1), \ldots, u_{a_{n-1}}(M_{n-1})]$$

Then the logarithm of the characteristic distribution function

$$\psi_{M_0, M_1, \ldots, M_{n-1}}(\theta_{a_0}, \theta_{a_1}, \ldots, \theta_{a_{n-1}}) =$$
$$= \ln \left\{ \int \exp \left(-i \sum_{k=0}^{n-1} u_{a_k} \theta_{a_k} \right) f_{a_0, a_1, \ldots, a_{n-1}} du_{a_0} du_{a_1}, \ldots, du_{a_{n-1}} \right\}$$

admits an expansion in Taylor series with respect to $\theta_{a_0}, \theta_{a_1}, \ldots, \theta_{a_{n-1}}$ with coefficients, the so-called semi-invariants [3]

$$S^{(n)}_{a_0, a_1, \ldots, a_{n-1}} = (-i)^n \frac{\partial^n}{\partial \theta_{a_0} \partial \theta_{a_1}, \ldots, \partial \theta_{a_{n-1}}} \psi_{a_0, a_1, \ldots, a_{n-1}} \Big|_{\theta_{a_0} = \theta_{a_1} = \ldots \theta_{a_{n-1}} = 0}$$

The semi-invariants are expressed in terms of the correlation tensors [3]:

$$S^{(2)}_{a_0 a_1} = T^{(2)}_{a_0 a_1}, \qquad S^{(3)}_{a_0 a_1 a_2} = T^{(3)}_{a_0 a_1 a_2}$$
$$S^{(4)}_{a_0 a_1 a_2 a_3} = T^{(4)}_{a_0 a_1 a_2 a_3} - \left(T^{(2)}_{a_0 a_1} T^{(2)}_{a_2 a_3} + T^{(2)}_{a_0 a_2} T^{(2)}_{a_1 a_3} + T^{(2)}_{a_0 a_3} T^{(2)}_{a_1 a_2} \right)$$
$$S^{(5)}_{a_0 a_1 a_2 a_3 a_4} = T^{(5)}_{a_0 a_1 a_2 a_3 a_4} - \left(T^{(2)}_{a_0 a_1} T^{(3)}_{a_2 a_3 a_4} + T^{(2)}_{a_0 a_2} T^{(3)}_{a_1 a_3 a_4} + T^{(2)}_{a_0 a_3} T^{(3)}_{a_1 a_2 a_4} + T^{(2)}_{a_0 a_4} T^{(3)}_{a_1 a_2 a_3} + \right.$$
$$\left. + T^{(2)}_{a_1 a_2} T^{(3)}_{a_0 a_3 a_4} + T^{(2)}_{a_1 a_3} T^{(3)}_{a_0 a_2 a_4} + T^{(2)}_{a_1 a_4} T^{(3)}_{a_0 a_2 a_3} + T^{(2)}_{a_2 a_3} T^{(3)}_{a_0 a_1 a_4} + T^{(2)}_{a_2 a_4} T^{(3)}_{a_0 a_1 a_3} + T^{(2)}_{a_3 a_4} T^{(3)}_{a_0 a_1 a_2} \right)$$

For a normal Gaussian distribution the semi-invariants of higher than third order vanish.

Since according to experimental results the actual distribution is close to Gaussian [1], the semi-invariants of higher than third order are small functions.

According to [3] the correlation tensors can be represented as

$$T^{(n)}_{a_0, a_1, \ldots, a_{n-1}} = S^{(n)}_{a_0, a_1, \ldots, a_{n-1}} + T^{(n)g}_{a_0, a_1, \ldots, a_{n-1}} \qquad \text{for } n > 3$$

where $T^{(n)g}_{a_0, a_1, \ldots, a_{n-1}}$ are the correlation tensors of the Gaussian distribution.

We note that for $r_k \to \infty$, $T \to T^g$ and therefore $S \to 0$, and the classical Fourier transformation is applicable. Then the Fourier transform of the correlation tensors of rank higher than third is the sum of two Fourier transforms: The classical $\Phi(S)$, and the generalized Fourier transform of the Gaussian distribution $\Phi(T^g)$. The latter is very easily introduced using $\delta(k)$, the Dirac delta function, with the use of the following properties of the Fourier transform

$$\frac{1}{(2\pi)^2}\int f(r_1-r_2)\exp\left[-i(k_1r_1+k_2r_2)\right]dr_1dr_2=\varphi(k_1)\,\delta(k_1+k_2) \qquad (10)$$

$$\frac{1}{(2\pi)^2}\int f_1(r_1)\,f_2(r_1-r_2)\exp\left[-i(k_1r_1+k_2r_2)\right]dr_1dr_2=\varphi_1(k_1+k_2)\,\varphi_2(k_2) \qquad (11)$$

$$\frac{1}{2\pi}\int \exp(-ikr)\,dr=\delta(k), \qquad k\delta(k)=0 \qquad (12)$$

The spectral function of fourth rank is expressed as

$$\begin{aligned}\tau^{(4)}_{a_0a_1a_2a_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)&=\tau^{(2)}_{a_0a_1}(\mathbf{k}_1)\,\tau^{(2)}_{a_2a_3}(\mathbf{k}_3)\,\delta(\mathbf{k}_2+\mathbf{k}_3)+\\&+\tau^{(2)}_{a_0a_2}(\mathbf{k}_2)\,\tau^{(2)}_{a_1a_3}(\mathbf{k}_3)\,\delta(\mathbf{k}_1+\mathbf{k}_3)+\tau^{(2)}_{a_0a_3}(\mathbf{k}_3)\,\tau^{(2)}_{a_1a_2}(\mathbf{k}_2)\,\delta(\mathbf{k}_2+\mathbf{k}_1)\end{aligned} \qquad (13)$$

The spectral tensor of arbitrary rank can be introduced using the recurrence relation [3]:

$$\begin{aligned}T^{(n)\,g}_{a_0,\ldots,a_{n-1}}&=T^{(n-2)\,g}_{a_0,a_1,\ldots,a_{n-3}}\,T^{(2)}_{a_{n-2}a_{n-1}}+\\&+T^{(n-2)\,g}_{a_0a_1,\ldots,a_{n-4}a_{n-2}}\,T^{(2)}_{a_{n-3}a_{n-1}}+\ldots+T^{(n-2)\,g}_{a_1a_2,\ldots,a_{n-2}}\,T^{(2)}_{a_0a_{n-1}}\end{aligned}$$

The examination carried out justifies the application of the method, and permits an explicit expression to be obtained for the initial spectral tensors of a given Gaussian distribution.

We note that such a choice of initial conditions imposes no essential limitation on the generality of the problem investigated, because in the solution obtained:

a) the spectral tensors of third order are different from zero for $t > 0$;

b) the solution takes dissipation into account, which acts chiefly on the small-scale turbulence and leads to departures from the Gaussian distribution in the course of time.

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